B.Sc. Part III 7th Paper

Diff. Eqns (contd.)

Q. Solve

$$\frac{d^2y}{dx^2} - \cot x \frac{dy}{dx} - (1-\cot x)\, y = e^x \sin x$$

Soln The given equation is of the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

Here, $P = -\cot x$, $Q = -(1-\cot x)$, $R = e^x \sin x$ —(1)

$\therefore$ $1+P+Q = 1-\cot x - (1-\cot x) = 0$

$\Rightarrow$ e^x is a part of CF of the given eqn.

Let $u = e^x$ and $y = uv$ is the general soln of the given eqn.

$\therefore$ v is given by $\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$

$\Rightarrow$ $\frac{d^2v}{dx^2} + \left(-\cot x + \frac{2}{e^x} e^x\right)\frac{dv}{dx} = \frac{e^x \sin x}{e^x}$

$\Rightarrow$ $\frac{d^2v}{dx^2} + (2-\cot x)\frac{dv}{dx} = \sin x$

Put $\frac{dv}{dx} = z$

$\Rightarrow$ $\frac{dz}{dx} + (2-\cot x) z = \sin x$, which is a linear eqn.

$\therefore$ I.F. $= e^{\int(2-\cot x)dx} = e^{2x - \log \sin x} = e^{2x} \cdot e^{\log(\sin x)^{-1}}$

$= e^{2x} \cdot \frac{1}{\sin x}$.

Soln of linear eqn is

$$Z \times IF = \int \sin x \cdot IF \, dx$$

$$\Rightarrow Z \times \frac{e^{2x}}{\sin x} = \int \sin x \cdot \frac{e^{2x}}{\sin x} dx$$

$$\Rightarrow Z \frac{e^{2x}}{\sin x} = \frac{e^{2x}}{2} + K$$

$$\Rightarrow Z = \frac{\sin x}{2} + K \sin x \cdot e^{-2x}$$

$$\Rightarrow \frac{dv}{dx} = \frac{\sin x}{2} + K \sin x \cdot e^{-2x}$$

$$\Rightarrow dv = \frac{1}{2} \sin x \, dx + K \sin x \cdot e^{-2x} dx$$

integrating, we get

$$\Rightarrow V = \frac{1}{2} \int \sin x \, dx + K \int e^{-2x} \sin x \, dx$$

But $\displaystyle \int e^{ax} \sin bx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx)$

$$\Rightarrow \int e^{-2x} \sin x \, dx = \frac{e^{-2x}}{4+1} (-2 \sin x - \cos x)$$

$$= \frac{e^{-2x}}{-5} (\cos x + 2 \sin x)$$

$$\Rightarrow V = -\frac{1}{2} \cos x - \frac{e^{-2x} \cdot K}{5} (\cos x + 2 \sin x) + K_1 \longrightarrow (A)$$

Hence $y = uv$ is the complete soln

where $u = e^{x}$ and v is given by eq (A)